# The Fourth Dimension

# By Charles H. Hinton

# 1904

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## Four-Dimensional Space

There is nothing more indefinite, and at the same time more real, than that which we indicate when we speak of the "higher." In our social life we see it evidenced in a greater complexity of relations. But this complexity is not all. There is, at the same time, a contact with, an apprehension of, something more fundamental, more real.

With the greater development of man there comes a consciousness of something more than all the forms in which it shows itself. There is a readiness to give up all the visible and tangible for the sake of those principles and values of which the visible and tangible are the representation. The physical life of civilized man and of a mere savage are practically the same, but the civilized man has discovered a depth in his existence, which makes him feel that that which appears all to the savage is a mere externality and appurtenage to his true being.

Now, this higher--how shall we apprehend it? It is generally embraced by our religious faculties, by our idealizing tendency. But the higher existence has two sides. It has a being as well as qualities. And in trying to realize it through our emotions we are always taking the subjective view. Our attention is always fixed on what we feel, what we think. Is there any way of apprehending the higher after the purely objective method of a natural science? I think that there is.

Plato, in a wonderful allegory, speaks of some men living in such a condition that they were practically reduced to be the denizens of a shadow world. They were chained, and perceived but the shadows of themselves and all real objects projected on a wall, towards which their faces were turned. All movements to them were but movements on the surface, all shapes but the shapes of outlines with no substantiality.

Plato uses this illustration to portray the relation between true being and the illusions of the sense world. He says that just as a man liberated from his chains could learn and discover that the world was solid and real, and could go back and tell his bound companions of this greater higher reality, so the philosopher who has been liberated, who has gone into the thought of the ideal world, into the world of ideas greater and more real than the things of sense, can come and tell his fellow men of that which is more true than the visible sun--more noble than Athens, the visible state.

Now, I take Plato's suggestion; but literally, not metaphorically. He imagines a world which is lower than this world, in that shadow figures and shadow motions are its constituents; and to it he contrasts the real world. As the real world is to this shadow world, so is the higher world to our world. I accept his analogy. As our world in three dimensions is to a shadow or plane world, so is the higher world to our three-dimensional world. That is, the higher world is four-dimensional; the higher being is, so far as its existence is concerned apart from its qualities, to be sought through the conception of an actual existence spatially higher than that which we realize with our senses.

Here you will observe I necessarily leave out all that gives its charm and interest to Plato's writings. All those conceptions of the beautiful and good which live immortally in his pages.

All that I keep from his great storehouse of wealth is this one thing simply--a world spatially higher than this world, a world which can only be approached through the stocks and stones of it, a world which must be apprehended laboriously, patiently, through the material things of it, the shapes, the movements, the figures of it.

We must learn to realize the shapes of objects in this world of the higher man; we must become familiar with the movements that objects make in his world, so that we can learn something about his daily experience, his thoughts of material objects, his machinery.

The means for the prosecution of this enquiry are given in the conception of space itself.

It often happens that that which we consider to be unique and unrelated gives us, within itself, those relations by means of which we are able to see it as related to others, determining and determined by them.

Thus, on the earth is given that phenomenon of weight by means of which Newton brought the earth into its true relation to the sun and other planets. Our terrestrial globe was determined in regard to other bodies of the solar system by means of a relation which subsisted on the earth itself.

And so space itself bears within it relations of which we can determine it as related to other space. For within space are given the conceptions of point and line, line and plane, which really involve the relation of space to a higher space.

Where one segment of a straight line leaves off and another begins is a point, and the straight line itself can be generated by the motion of the point.

One portion of a plane is bounded from another by a straight line, and the plane itself can be generated by the straight line moving in a direction not contained in itself.

Again, two portions of solid space are limited with regard to each other by a plane; and the plane, moving in a direction not contained in itself, can generate solid space.

Thus, going on, we may say that space is that which limits two portions of higher space from each other, and that our space will generate the higher space by moving in a direction not contained in itself.

Another indication of the nature of four-dimensional space can be gained by considering the problem of the arrangement of objects.

If I have a number of swords of varying degrees of brightness, I can represent them in respect of this quality by points arranged along a straight line.

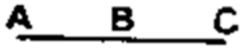


FIGURE 22

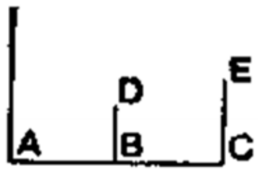


FIGURE 23

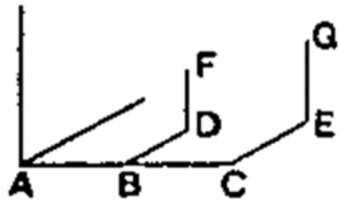


FIGURE 24

If I place a sword at A, figure 22, and regard it as having a certain brightness, then the other swords can be arranged in a series along the line, as at A, B, C, etc., according to their degrees of brightness.

If now I take account of another quality, say length, they can be arranged in a plane. Starting from A, B, C, I can find points to represent different degrees of length along such lines as AF, BD, CE, drawn from A and B and C (see fig. 23). Points on these lines represent different degrees of length with the same degree of brightness. Thus the whole plane is occupied by points representing all conceivable varieties of brightness and length.

Bringing in a third quality, say sharpness, I can draw, as in figure 24, any number of upright lines. Let distances along these upright lines represent degrees of sharpness, thus the points F and G will represent swords of certain definite degrees of the three qualities mentioned, and the whole of space will serve to represent all conceivable degrees of these three qualities.

If now I bring in a fourth quality, such as weight, and try to find a means of representing it as I did the other three qualities, I find a difficulty. Every point in space is taken up by some conceivable combination of the three qualities already taken.

To represent four qualities in the same way as that in which I have represented three, I should need another dimension of space.

Thus we may indicate the nature of four-dimensional space by saying that it is a kind of space which would give positions representative of four qualities, as three-dimensional space gives positions representative of three qualities.

# A Chapter in the History of Four Space

Parmenides, and the Asiatic thinkers with whom he is in close affinity, propound a theory of existence which is in close accord with a conception of a possible relation between a higher and a lower dimensional space. This theory, prior and in marked contrast to the main stream of thought, which we shall afterwards describe, forms a closed circle by itself. It is one which in all ages has had a strong attraction for pure intellect, and is the natural mode of thought for those who refrain from projecting their own volition into nature under the guise of causality.

According to Parmenides of the school of Flea, the all is one, unmoving and unchanging. The permanent amid the transient--that foothold for thought, that solid ground for feeling, on the discovery of which depends all our life--is no phantom; it is the image amidst deception of true being, the eternal, the unmoved, the one. Thus says Parmenides.

But how explain the shifting scene, these mutations of things!

"Illusion," answers Parmenides. Distinguishing between truth and error, he tells of the true doctrine of the one--the false opinion of a changing world. He is no less memorable for the manner of his advocacy than for the cause he advocates. It is as if from his firm foothold of being he could play with the thoughts under the burden of which others labored, for from him springs that fluency of supposition and hypothesis which forms the texture of Plato's dialectic.

Can the mind conceive a more delightful intellectual picture than that of Parmenides, pointing to the one, the true, the unchanging, and yet on the other hand ready to discuss all manner of false opinion, forming a cosmogony too, false "but mine own" after the fashion of the time?

In support of the true opinion he proceeded by the negative way of showing the self-contradictions in the ideas of change and motion. It is doubtful if his criticism, save in minor points, has ever been successfully refuted. To express his doctrine in the ponderous modern way we must make the statement that motion is phenomenal not real.

Let us represent his doctrine.

Imagine a sheet of still water into which a slanting stick is being lowered with a motion vertically downwards. Let 1, 2, 3 (fig. 25), be three consecutive positions of the stick. A, B, C, will be three consecutive positions of the meeting of the stick with the surface of the water. As the stick passes down, the meeting will move from A on to B and C.

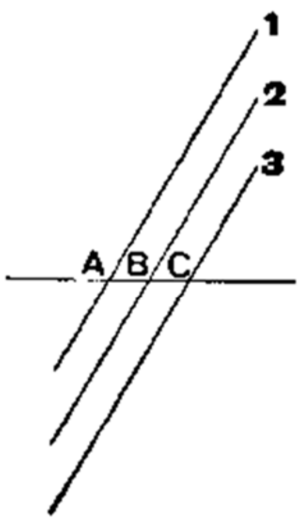


FIGURE 25

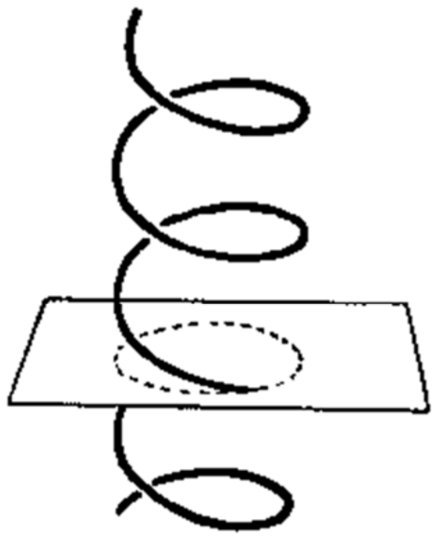

FIGURE 26

Suppose now all the water to be removed except a film. At the meeting of the film and the stick there will be an interruption of the film. If we suppose the film to have a property, like that of a soap bubble, of closing up round any penetrating object then as the stick goes vertically downwards the interruption in the film will move on.

If we pass a spiral through the film, the intersection will give a point moving in a circle shown by the dotted lines (fig. 26). Suppose now the spiral to be still and the film to move vertically upwards the whole spiral will be represented in the film of the consecutive positions of the point of intersection. In the film the permanent existence of the spiral is experienced as a time series--the record of traversing the spiral is a point moving in a circle. If now we suppose a consciousness connected with the film in such a way that the intersection of the spiral with the film gives rise to a conscious experience, we see that we shall have in the film a point moving in a circle, conscious of its motion, knowing nothing of that real spiral the record of the successive intersections of which by the film is the motion of the point.

It is easy to imagine complicated structures of the nature of the spiral, structures consisting of filaments, and to suppose also that these structures are distinguishable from each other at every section. If we consider the intersections of these filaments with the film as it passes to be the atoms constituting a filmar universe, we shall have in the film a world of apparent motion; we shall have bodies corresponding to the filamentary structure, and the positions of these structures with regard to one another will give rise to bodies in the film moving amongst one another. This mutual motion is apparent merely. The reality is of permanent structures stationary, and all the relative motions accounted for by one steady movement of the film as a whole.

Thus we can imagine a plane world, in which all the variety of motion is the phenomenon of structures consisting of filamentary atoms traversed by a plane of consciousness. Passing to four dimensions and our space, we can conceive that all things and movements in our world are the reading off of a permanent reality by a space of consciousness. Each atom at every moment is not what it was, but a new part of that endless line which is itself. And all this system successively revealed in the time which is but the succession of consciousness, separate as it is in parts, in its entirety is one vast unity. Representing Parmenides' doctrine thus, we gain a firmer hold on it than if we merely let his words rest, grand and massive, in our minds. And we have gained the means also of representing phases of that Eastern thought to which Parmenides was no stranger. Modifying his uncompromising doctrine, let us suppose, to go back to the plane of consciousness and the structure of filamentary atoms, that these structures are themselves moving--are acting, living. Then, in the transverse motion of the film, there would be two phenomena of motion, one due to the reading off in the film of the permanent existences as they are in themselves, and another phenomenon of motion due to the modification of the record of the things themselves, by their proper motion during the process of traversing them.

Thus a conscious being in the plane would have, as it were, a twofold experience. In the complete traversing of the structure, the Intersection of which with the film gives his conscious all, the main and principal movements and actions which he went through would be the record of his higher self as it existed unmoved and unacting. Slight modifications and deviations from these movements and actions would represent the activity and self-determination of the complete being, of his higher self.

It is admissible to suppose that the consciousness in the plane has a share in that volition by which the complete existence determines itself. Thus the motive and will, the initiative and life, of the higher being, would be represented in the case of the being in the film by an initiative and a will capable, not of determining any great things or important movements in his existence, but only of small and relatively insignificant activities. In all the main features of his life his experience would be representative of one state of the higher being whose existence determines his as the film passes on. But in his minute and apparently unimportant actions he would share in that will and determination by which the whole of the being he really is acts and lives.

An alteration of the higher being would correspond to a different life history for him. Let us now make the supposition that film after film traverses these higher structures, that the life of the real being is read off again and again in successive waves of consciousness. There would be a succession of lives in the different advancing planes of consciousness, each differing from the preceding, and differing in virtue of that will and activity which in the preceding had not been devoted to the greater and apparently most significant things in life, but the minute and apparently unimportant. In all great things the being of the film shares in the existence of his higher self as it is at any one time. In the small things he shares in that volition by which the higher being alters and changes, acts and lives.

Thus we gain the conception of a life changing and developing as a whole, a life in which our separation and cessation and fugitiveness are merely apparent, but which in its events and course alters, changes, develops; and the power of altering and changing this whole neé lies in the will and power the limited being has of directing, guiding, altering himself in the minute things of his existence.

Transferring our conceptions to those of an existence in a higher dimensionality traversed by a space of consciousness, we have an illustration of a thought which has found frequent and varied expression. When, however, we ask ourselves what degree of truth there lies in it, we must admit that, as far as we can see, it is merely symbolical. The true path in the investigation of a higher dimensionality lies in another direction.

The significance of the Parmenidean doctrine lies in this: that here, as again and again, we find that those conceptions which man introduces of himself, which he does not derive from the mere record of his outward experience, have a striking and significant correspondence to the conception of a physical existence in a world of a higher space. How close we come to Parmenides' thought by this manner of representation it is impossible to say. What I want to point out is the adequateness of the illustration, not only to give a static model of his doctrine, but one capable as it were, of a plastic modification into a correspondence into kindred forms of thought. Either one of two things must be true-that four-dimensional conceptions give a wonderful power of representing the thought of the East, or that the thinkers of the East must have been looking at and regarding four-dimensional existence.

And from the numerical idealism of Pythagoras there is but a step to the more rich and full idealism of Plato. That which is apprehended by the sense of touch we put as primary and real, and the other senses we say are merely concerned with appearances. But Plato took them all as valid, as giving qualities of existence. That the qualities were not permanent in the world as given to the senses forced him to attribute to them a different kind of permanence. He formed the conception of a world of ideas, in which all that really is, all that affects us and gives the rich and wonderful wealth of our experience, is not fleeting and transitory, but eternal. And of this real and eternal we see in the things about us the fleeting and transient images.

And this world of ideas was no exclusive one, wherein was no place for the innermost convictions of the soul and its most authoritative assertions. Therein existed justice beauty-the one, the good, all that the soul demanded to be. The world of ideas, Plato's wonderful creation preserved for man, for his deliberate investigation and their sure development, all that the rude incomprehensible changes of a harsh experience scatters and destroys.

Plato believed in the reality of ideas. He meets us fairly and squarely. Divide a line into two parts, he says (fig. 27); one to represent the real objects in the world, the other to represent the transitory appearances, such as the image in still water, the glitter of the sun on a bright surface, the shadows on the clouds.

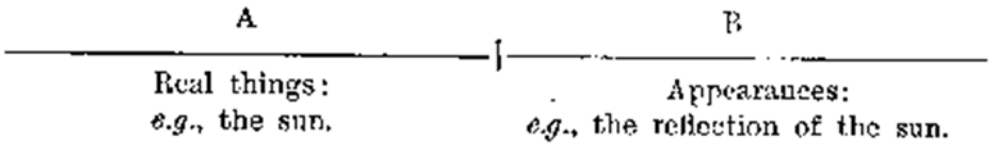


FIGURE 27

Take another line and divide it into two parts (fig. 28), one representing our ideas, the ordinary occupants of our minds, such as whiteness, equality, and the other representing our true knowledge, which is of eternal principles, such as beauty, goodness.

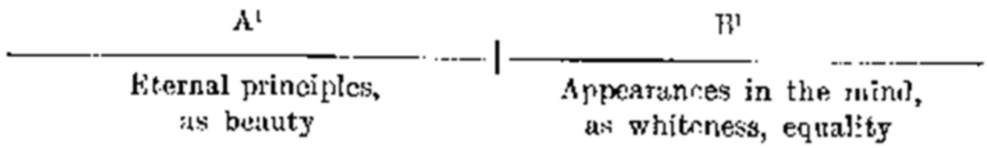


FIGURE 28

Then as A is to B, so is A' to B'.

That is, the soul can proceed, going away from real things to a region of perfect certainty, where it beholds what is, not the scattered reflections; beholds the sun, not the glitter on the sands; true being, not chance opinion.

Now, this is to us, as it was to Aristotle, absolutely inconceivable from a scientific point of view. We can understand that a being is known in the fullness of his relations; it is in his relations to his circumstances that a man's character is known; it is in his acts under his conditions that his character exists. We cannot grasp or conceive any principle of individuation apart from the fullness of the relations to the surroundings.

But suppose now that Plato is talking about the higher man--the four-dimensional being that is limited in our external experience to a three-dimensional world. Do not his words begin to have a meaning? Such a being would have a consciousness of motion which is not as the ! │ ┐ motion he can see with the eyes of the body. He, in his own being, knows a reality to which the outward matter of this too solid earth is flimsy superficiality. He too knows a mode of being, the fullness of relations, in which can only be represented in the limited world of sense, as the painter unsubstantially portrays the depths of woodland, plains, and air. Thinking of such a being in man, was not Plato's line well divided?

It is noteworthy that, if Plato omitted his doctrine of the independent origin of ideas, he would present exactly the four-dimensional argument; a real thing as we think it is an idea. A plane being's idea of a square object is the idea of an abstraction, namely, a geometrical square. Similarly our idea of a solid thing is an abstraction, for in our idea there is not the four-dimensional thickness which is necessary, however slight, to give reality. The argument would then run, as a shadow is to a solid object, so is the solid object to the reality. Thus A and B' would be identified.

In the allegory which I have already alluded to, Plato in almost as many words shows forth the relation between existence in a superficies and in solid space. And he uses this relation to point to the conditions of a higher being.

He imagines a number of men prisoners, chained so that they look at the wall of a cavern in which they are confined, with their backs to the road and the light. Over the road pass men and women, figures and processions, but of all this pageant all that the prisoners behold is the shadow of it on the wall whereon they gaze. Their own shadows and the shadows of the things in the world are all that they see, and identifying themselves with their shadows related as shadows to a world of shadows, they live in a kind of dream.

Plato imagines one of their number to pass out from amongst them into the real space world, and then returning to tell them of their condition.

Here he presents most plainly the relation between existence in a plane world and existence in a three-dimensional world. And he uses this illustration as a type of the manner in which we are to proceed to a higher state from the three-dimensional life we know.

It must have hung upon the weight of a shadow which path he took! Whether the one we shall follow toward the higher solid and the four-dimensional existence, or the one which makes ideas the higher realities, and the direct perception of them the contact with the truer world.

# Metageometry

The theories which are generally connected with the names of Lobatchewsky and Bolyai bear a singular and curious relation to the subject of higher space.

In order to show what this relation is, I must ask the reader to be at the pains to count carefully the sets of points by which I shall estimate the volumes of certain figures.

No mathematical processes beyond this simple one of counting will be necessary.

Let us suppose we have before us in figure 29 a plane covered with points at regular intervals, so placed that every four determine a square.

Now it is evident that as four points determine a square, so four squares meet in a point.

Thus, considering a point inside a square as belonging to it, we may say that a point on the corner of a square belongs to it and to four others equally: belongs a quarter of it to each square.

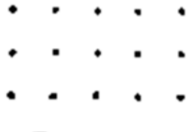

Figure 29

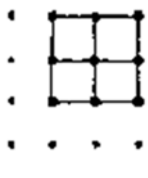

Figure 30

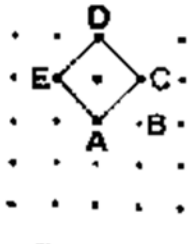


Figure 31

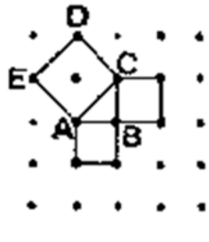


Figure 32

Thus the square ACDE (fig. 31) contains one point, and has four points at the four corners. Since one-fourth of each of these four belongs to the square, the four together count as one point, and the point value of the square is two points--the one inside and the four at the corner make two points belonging to it exclusively.

Now the area of this square is two unit squares, as can be seen by drawing two diagonals in figure 32.

We also notice that the square in question is equal to the sum of the squares on the sides AB, BC, of the right-angled triangle ABC. Thus we recognize the proposition that the square on the hypotenuse is equal to the sum of the squares on the two sides of a right-angled triangle.

Now suppose we set ourselves the question of determining whereabouts, in the ordered system of points, the end of a line would wE┐ come when it turned about a point keeping one extremity fixed at the point.

We can solve this problem in a particular case. If we can find a square lying slantwise amongst the dots which is equal to one which goes regularly, we shall know that the two sides are equal, and that the slanting side is equal to the straight-way side. Thus the volume and shape of a figure remaining unchanged will be the test of its having rotated about the point, so that we can say that its side in its first position would turn into its side in the second position.

Now, such a square can be found in the one whose side is five units in length.

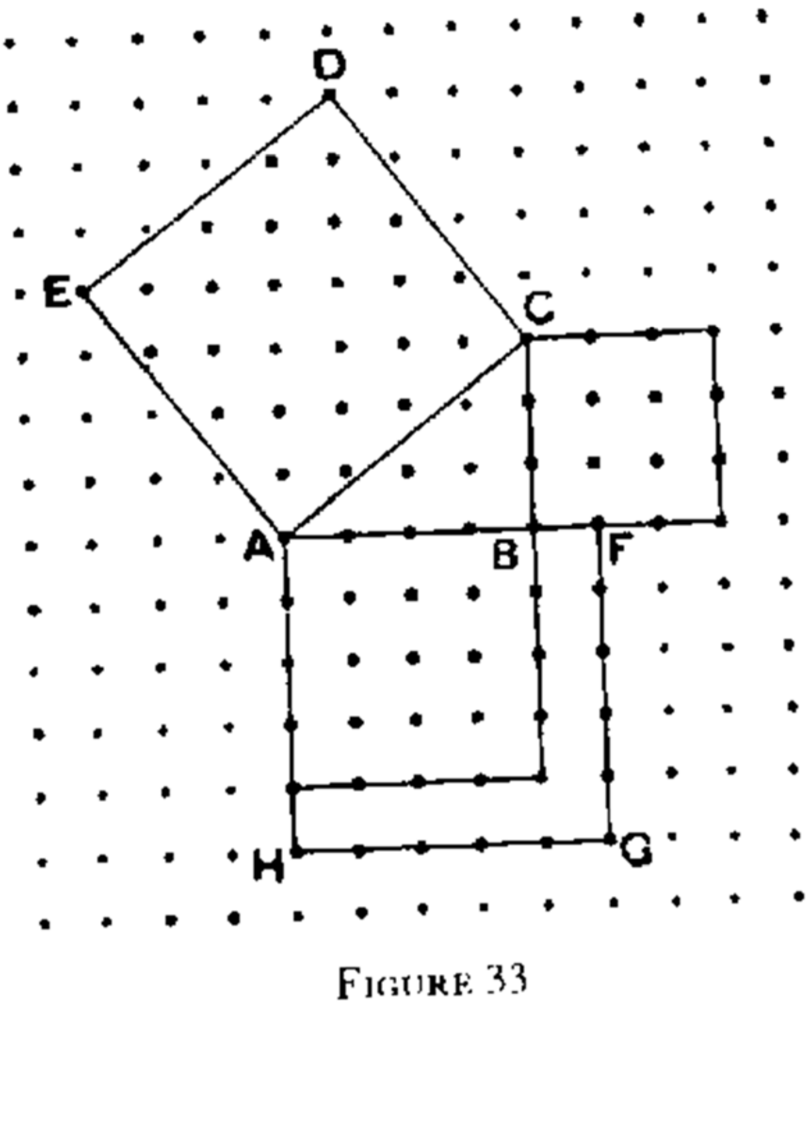


FIGURE 33

In figure 33, in the square on AB, there are

| | |
|---|---|
| 9 points interior | 9 |
| 4 at the corners | 1 |
| 4 sides with 3 on each side, considered as 1 1/2, on each side, because belonging equally to two squares | 6 |

The total is 16. There are 9 points in the square on BC. In the square on AC there are--

| | |
|---|---|
| 24 points inside | 24 |
| 4 at the corners | 1 |

or 25 altogether.

Hence we see again that the square on the hypotenuse is equal to the squares on the sides.

Now take the square AFHC, which is larger than the square on AB. It contains 25 points.

| | |
|---|---|
| 16 inside | 16 |
| 16 on the sides, counting as | 8 |
| 4 on the corners | 1 |

making 25 altogether.

If two squares are equal we conclude the sides are equal. Hence, the line AF turning round A would move so that it would after a certain turning coincide with AC.

This is preliminary, but it involves all the mathematical difficulties that will present themselves.

There are two alterations of a body by which its volume is not changed.

One is the one we have just considered, rotation, the other is what is called shear.

Consider a book, or heap of loose pages. They can be slid so that each one slips over the preceding one, and the whole assumes the shape b in figure 34.

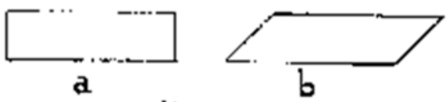


Figure 34

The deformation is not shear alone, but shear accompanied by rotation.

Shear can be considered as produced in another way.

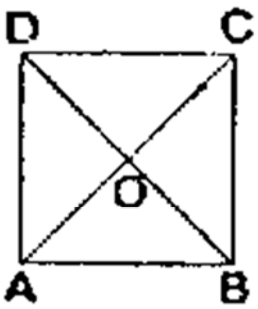


FIGURE 35

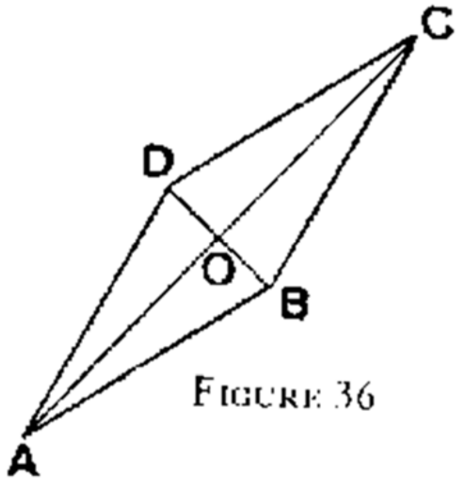


FIGURE 36

Take the square ABCD (fig. 35), and suppose that it is pulled out from along one of its diagonals both ways, and proportionately compressed along the other diagonal. It will assume the shape in figure 36.

This compression and expansion along two lines at right angles is what is called shear; it is equivalent to the sliding illustrated above combined with a turning round.

In pure shear a body is compressed and extended in two directions at right angles to each other, so that its volume remains unchanged.

Now we know that our material bodies resist shear--shear does violence to the internal arrangement of their particles, but they turn as wholes without such internal resistance.

But there is an exception. In a liquid shear and rotation take place equally easily, there is no more resistance against a shear than there is against a rotation.

Now, suppose all bodies were to be reduced to the liquid state, in which they yield to shear and to rotation equally easily, and then were to be reconstructed as solids, but in such a way that shear and rotation had interchanged places.

That is to say, let us suppose that when they had become solids again they would shear without offering any internal resistance, but a rotation would do violence to their internal arrangement.

That is, we should have a world in which shear would have taken the place of rotation.

A shear does not alter the volume of a body: thus an inhabitant living in such a world would look on a body sheared as we look on a body rotated. He would say that it was of the same shape, but had turned a bit round.

Let us imagine a Pythagoras in this world going to work to investigate, as is his wont.

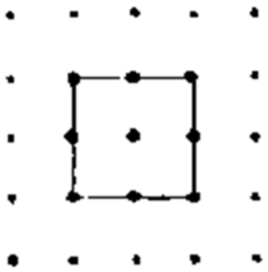

FIGURE 37

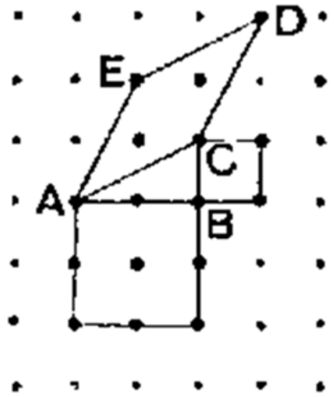


FIGURE 38

Figure 37 represents a square unsheared. Figure 38 represents a square sheared. It is not the figure into which the square in figure 37 would turn, but the result of shear on some square not drawn. It is a simple slanting placed figure, taken now as we took a simple slanting placed square before. Now, since bodies in this world of shear offer no internal resistance to shearing, and keep their volume when sheared, an inhabitant accustomed to them would not consider that they altered their shape under shear. He would call ACDE as much a square as the square in figure 37. We will call such figures shear squares. Counting the dots in ACDF, we find

| | |
|---|---|
| 2 inside | 2 |
| 4 at corners | 1 |

or a total of 3.

Now, the square on the side AB has 4 points, that on BC has 1 point. Here the shear square on the hypotenuse has not 5 points but 3; it is not the sum of the squares on the sides, but the difference.

This relation always holds. Look at figure 39.

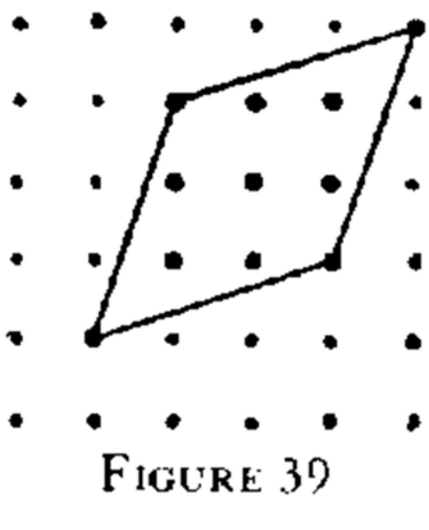

FIGURE 39

Shear square on hypotenuse

| | |
|---|---|
| 7 internal | 7 |
| 4 at corners | 1 |
| | — |
| | 8 |

Square on one side--which the reader can draw for himself--

| | |
|---|---|
| 4 internal | 4 |
| 8 on sides | 4 |
| 4 at corners | 1 |
| | — |
| | 9 |

The square on the other side is 1. Hence in this case again the difference is equal to the shear square on the hypotenuse, 9 - 1 = 8.

Thus in a world of shear the square on the hypotenuse would be equal to the difference of the squares on the sides of a right-angled triangle.

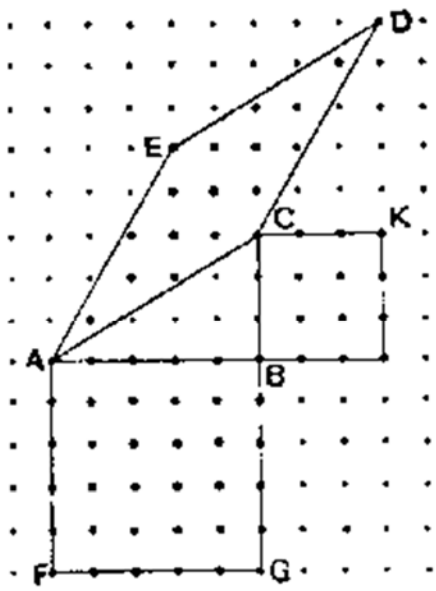


FIGURE 40

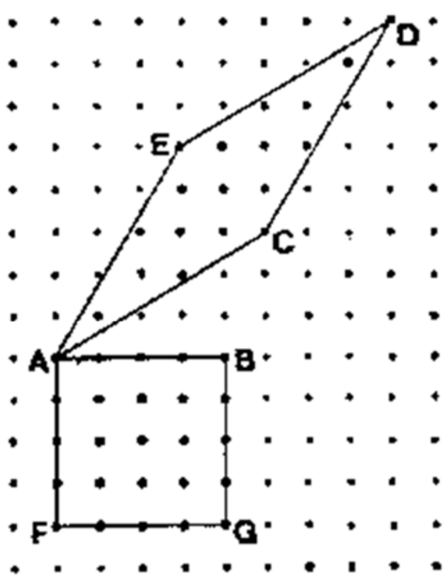


FIGURE 41

In figure 40 another shear square is drawn on which the above relation can be tested.

What now would be the position a line on turning by shear would take up?

We must settle this in the same way as previously with our turning.

Since a body sheared remains the same, we must find two equal bodies, one in the straight way, one in the slanting way, which have the same volume. Then the side of one will by turning become the side of the other, for the two figures are each what the other becomes by a shear turning.

We can solve the problem in a particular case--

In the figure ACDE (fig. 41) there are

```
15 inside       15

4 at corners     1
```

a total of 16.

Now in the square ABCF, there are 16--

```
9 inside         9

12 on sides      6

4 at corners     1

                ——

                16
```

Hence the square on AB would, by the shear turning, become the shear square ACDE.

And hence the inhabitant of this world would say that the line AB turned into the line AC. These two lines would be to him two lines of equal length, one turned a little way round from the other.

That is, putting shear in place of rotation, we get a different kind of figure, as the result of the shear rotation, from what we got with our ordinary rotation. And as a consequence we get a position for the end of a line of invariable length when it turns by the shear rotation, different from the position which it would assume on turning by our rotation.

A real material rod in the shear world would, on turning about A, pass from the position AB to the position AC. We say that its length alters when it becomes AC, but this transformation of AB would seem to an inhabitant of the shear world like a turning of AB without altering in length.

If now we suppose a communication of ideas that takes place between one of ourselves and an inhabitant of the shear world, there would evidently be a difference between his views of distance and ours.

We should say that his line AB increased in length in turning to AC. He would say that our line AF (fig. 33) decreased in length in turning to AC. He would think that what we called an equal line was in reality a shorter one.

We should say that a rod turning round would have its extremities in the positions we call at equal distances. So would he--but the positions would be different. He could, like us, appeal to the properties of matter. His rod to him alters as little as ours does to us.

Now, is there any standard to which we could appeal, to say which of the two is right in this argument? There is no standard.

We should say that, with a change of position, the configuration and shape of his objects altered. He would say that the configuration and shape of our objects altered in what we called merely a change of position. Hence distance independent of position is inconceivable, or practically, distance is solely a property of matter.

There is no principle to which either party in this controversy could appeal. There is nothing to connect the definition of distance with our ideas rather than with his, except the behavior of an actual piece of matter. For the study of the processes which go on in our world the definition of distance given by taking the sum of the squares is of paramount importance to us. But as a question of pure space without making any unnecessary assumptions, the shear world is just as possible and just as interesting as our world.

It was the geometry of such conceivable worlds that Lobatchewsky and Bolyai studied.

This kind of geometry has evidently nothing to do directly with four-dimensional space.

But a connection arises in this way. It is evident that, instead of taking a simple shear as I have done, and defining it as that change of the arrangement of the particles of a solid which they will undergo without offering any resistance due to their mutual action, I might take a complex motion, composed of a shear and a rotation together, or some other kind of deformation.

Let us suppose such an alteration picked out and defined as the one which means simple rotation; then the type, according to which all bodies will alter by this rotation, is fixed.

Looking at the movements of this kind, we should say that the objects were altering their shape as well as rotating. But to the inhabitants of that world they would seem to be unaltered, and our figures in their motions would seem to them to alter.

In such a world the features of geometry are different. We have seen one such difference in the case of our illustration of the world of shear, where the square on the hypotenuse was equal to the difference, not the sum, of the squares on the sides.

In our illustration we have the same laws of parallel lines as in our ordinary rotation world, but in general the laws of parallel lines are different.

In one of these worlds of a different constitution of matter, through one point there can be two parallels to a given line, in another of them there can be none; that is, although a line be drawn parallel to another it will meet it after a time.

Now it was precisely in this respect of parallels that Lobatchewsky and Bolyai discovered these different worlds. They did not think of them as worlds of matter, but they discovered that space did not necessarily mean that our law of parallels is true. They made the distinction between laws of space and laws of matter, although that is not the form in which they stated their results.

The way in which they were led to these results was the following. Euclid had stated the existence of parallel lines as a postulate--putting frankly this unproved proposition--that one line and only one parallel to a given straight line can be drawn, as a demand, as something that must be assumed. The words of his ninth postulate are these: "if a straight line meeting two other straight lines makes the interior angles on the same side of it equal to two right angles, the two straight lines will never meet."

The mathematicians of later ages did not like this bald assumption, and not being able to prove the proposition they called it an axiom--the eleventh axiom.

Many attempts were made to prove the axiom; no one doubted of its truth, but no means could be found to demonstrate it. At last an Italian, Sacchieri, unable to find a proof, said: "Let us suppose it not true." He deduced the results of there being possibly two parallels to one given line through a given point, but feeling the waters too deep for the human reason, he devoted the latter half of his book to disproving what he had assumed in the first part.

Then Bolyai and Lobatchewsky with firm step entered on the forbidden path. There can be no greater evidence of the indomitable nature of the human spirit, or of its manifest destiny to conquer all those limitations which bind it down within the sphere of sense than this grand assertion of Bolyai and Lobatchewsky.

Take a line AB and a point C. We say and see and know that through C can only be drawn one line parallel to AB.

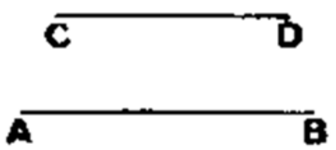

FIGURE 42

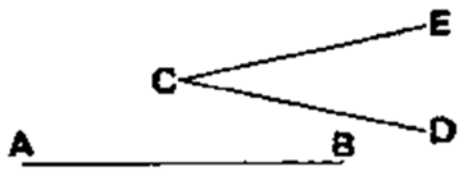

FIGURE 43

But Bolyai said: "I will draw two." Let CD be parallel to AB, that is, not meet AB however far produced, and let lines beyond CD also not meet AB; let there be a certain region between CD and CE, in which no line drawn meets AB. CE and CD produced backwards through C will give a similar region on the other side of C.

Nothing so triumphantly, one may almost say so insolently, ignoring of sense had ever been written before. Men had struggled against the limitations of the body, fought them, despised them, conquered them. But no one had ever thought simply as if the body, the bodily eyes, the organs of vision, all this vast experience of space, had never existed. The age-long contest of the soul with the body, the struggle for mastery, had come to a culmination. Bolyai and

Lobatchewsky simply thought as if the body was not. The struggle for dominion, the strife and combat of the soul were over; they had mastered, and the Hungarian drew his line.

Can we point out any connection, as in the case of Parmenides, between these speculations and higher space? Can we suppose it was any inner perception by the soul of a motion not known to the senses, which resulted in this theory so free from the bonds of sense? No such supposition appears to be possible.

Practically, however, metageometry had a great influence in bringing the higher space to the front as a working hypothesis. This can be traced to the tendency the mind has to move in the direction of least resistance. The results of the new geometry could not be neglected, the problem of parallels had occupied a place too prominent in the development of mathematical thought for its final solution to be neglected. But this utter independence of all mechanical considerations, this perfect cutting loose from the familiar intuitions, was so difficult that almost any other hypothesis was more easy of acceptance, and when Beltrami showed that the geometry of Lobatchewsky and Bolyai was the geometry of shortest lines drawn on certain curved surfaces, the ordinary definitions of measurement being retained, attention was drawn to the theory of a higher space. An illustration of Beltrami's theory is furnished by the simple consideration of hypothetical beings living on a spherical surface (fig. 44).

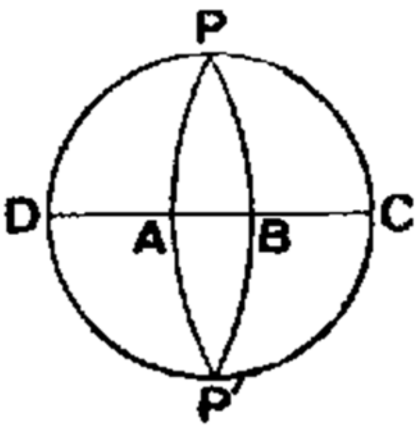


FIGURE 44

Let ABCD be the equator of a globe, and AP, BP, meridian lines drawn to the pole, P. The lines AB, AP, BP would seem to be perfectly straight to a person moving on the surface of the sphere, and unconscious of its curvature. Now AP and BP both make right angles with AB. Hence they satisfy the definition of parallels. Yet they meet in P. Hence a being living on a spherical surface, and unconscious of its curvature, would find that parallel lines would meet. He would also find that the angles in a triangle were greater than two right angles. In the triangle PAB, for instance, the angles at A and B are right angles, so the three angles of the triangle PAB are greater than two right angles.

Now in one of the systems of metageometry (for after Lobatchewsky had shown the way it was found that other systems were possible besides his), the angles of a triangle are greater than two right angles.

Thus a being on a sphere would form conclusions about his space which are the same as he would form if he lived on a plane, the matter in which had such properties as are presupposed by one of these systems of geometry. Beltrami also discovered a certain surface on which there could be drawn more than one "straight" line through a point which would not meet another given line. I use the word straight as equivalent to the line having the property of giving the shortest path between any two points on it. Hence, without giving up the ordinary methods of measurement, it was possible to find conditions in which a plane being would necessarily have an experience corresponding to Lobatchewsky's geometry. And by the consideration of a higher space, and a solid curved in such a higher space, it was possible to account for a similar experience in a space of three dimensions.

Now, it is far more easy to conceive of a higher dimensionality to space than to imagine that a rod in rotating does not move so that its end describes a circle. Hence, a logical conception having been found harder than that of a four-dimensional space, thought turned to the latter as a simple explanation of the possibilities to which Lobatchewsky had awakened it. Thinkers became accustomed to deal with the geometry of higher space--it was Kant, says Veronese, who first used the expression of "different spaces"--and with familiarity the inevitableness of the conception made itself felt.

From this point it is but a small step to adapt the ordinary mechanical conceptions to a higher spatial existence, and then the recognition of its objective existence could be delayed no longer. Here, too, as in so many cases, it turns out that the order and connection of our ideas is the order and connection of things.

What is the significance of Lobatchewsky's and Bolyai's work?

It must be recognized as something totally different from the conception of a higher space; it is applicable to spaces of any number of dimensions. By immersing the conception of distance in matter to which it properly belongs, it promises to be of the greatest aid in analysis; for the effective distance of any two particles is the product of complex material conditions and cannot be measured by hard and fast rules. Its ultimate significance is altogether unknown. It is a cutting loose from the bonds of sense, not coincident with the recognition of a higher dimensionality, but indirectly contributory thereto.

Thus, finally, we have come to accept what Plato held in the hollow of his hand; what Aristotle's doctrine of the relativity of substance implies. The vast universe, too, has its higher, and in recognizing it we find that the directing being within us no longer stands inevitably outside our systematic knowledge.